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Identity for Harmonic Oscillator Brackets

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An identity satisfied by the harmonic oscillator (Talmi-Moshinsky) brackets is derived from two equivalent methods for evaluating an integral often encountered in cluster model studies.

1. INTRODUCTION

The transformation coefficient for two particles with identical masses from their single particle coordinates to their relative and center-of-mass coordinates, in the harmonic oscillator basis, was first obtained by Talmi (1952). These harmonic oscillator brackets are denoted by: $\langle n_1 l_1, n_2 l_2, \lambda | nl, NL, \lambda\rangle$. Moshinsky (1959) derived an explicit formula for the special case $n_1 = n_2 = 0$ and showed how the general transformation brackets can be obtained from these by means of recurrence relations. Since then these transformation coefficients have been referred to in the literature as the Talmi-Moshinsky brackets (TMB). A general TMB formula for particles with unequal masses was derived by Smirnov (1961) and it is denoted by

$$B_\lambda(n_1 l_1, n_2 l_2, nl, NL) \equiv \langle n_1 l_1, n_2 l_2, \lambda | \mu_1 \mu_2 | nl, NL, \lambda\rangle$$
$$\equiv \langle n_1 l_1, n_2 l_2, \lambda | nl, NL, \lambda\rangle_D$$

where μ_1, μ_2 are the masses of the particles and $D = \mu_1/\mu_2$ is their ratio. More recently, several authors (Kumar, 1966; Baranger and Davies, 1966; Trlifaj, 1972; Dobes, 1977; Rashid, 1980) derived different expressions for the generalized TMB which are found to be useful and convenient in numerical computations (see, for instance, Sontona and Gmitro, 1972; Feng

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and Tamura, 1975). We derive here an identity satisfied by the general harmonic oscillator brackets for unequal masses.

2. THE INTEGRAL

Consider the particular integral,

$$I_{w_1,w_2,L}(p,q,s)=\int R^{w_1}S^{w_2}\exp[-pR^2+q\mathbf{R}\cdot\mathbf{S}-sS^2]$$
$$\times Y^*_{LM}(\mathbf{R}/R)\,Y_{LM}(\mathbf{S}/S)\,d\mathbf{R}\,d\mathbf{S} \qquad (1)$$

where $w_1=2n_1+L$ and $w_2=2n_2+L$. Such integrals are often encountered in the studies on the cluster model for nuclei (see, for instance, Wildermuth and McClure, 1966). Using the expansion (Magnus and Oberhettinger, 1954):

$$\exp(q\mathbf{R}\cdot\mathbf{S})=4\pi\left(\frac{\pi}{2qRS}\right)^{1/2}\sum_{l,m}I_{l+1/2}(qRS)\,Y^*_{lm}(\mathbf{R}/R)\,Y_{lm}(\mathbf{S}/S) \qquad (2)$$

where $I_{l+1/2}(qRS)$ is the Bessel function of half-integral order with imaginary argument $iqRS$ and then making use of Laplace transforms (Erdelyi et al., 1954), it is straightforward to obtain, after integration, the result:

$$I_{w_1,w_2,L}(p,q,s)=\pi^2q^L(2n_1+2L+1)!!(2n_2+2L+1)!!$$
$$\times[2^{n_1+n_2+2L+2}(2L+1)!!p^{(2n_1+2L+3)/2}s^{(2n_2+2L+3)/2}]$$
$$\times{}_2F_1\left(n_1+L+\frac{3}{2},n_2+L+\frac{3}{2};L+\frac{3}{2};\frac{q^2}{4ps}\right) \qquad (3)$$

We now evaluate the integral (1) in an alternative way, following in spirit the method of Kudeyarov *et al.* (1968). We first introduce the dimensionless coordinates $\mathbf{u}$ and $\mathbf{v}$:

$$\mathbf{u}=p^{1/2}\mathbf{R} \quad \text{and} \quad \mathbf{v}=s^{1/2}\mathbf{S}$$

Then, in order to separate the variables in the integral, we make the unitary transformations:

$$\mathbf{u}=\left(\frac{D}{1+D}\right)^{1/2}\mathbf{P}+\left(\frac{1}{1+D}\right)^{1/2}\mathbf{Q}$$
$$\mathbf{v}=\left(\frac{1}{1+D}\right)^{1/2}\mathbf{Q}-\left(\frac{D}{1+D}\right)^{1/2}\mathbf{P} \qquad (4)$$

The exponential factor in (1) now takes the separable form:

$$-aP^2-bQ^2 \qquad (5)$$

where

$$a=\frac{2D}{1+D}\left[1+\frac{q}{2(ps)^{1/2}}\right] \quad \text{and} \quad b=\frac{2}{1+D}\left[1-\frac{q}{2(ps)^{1/2}}\right]$$

We write for the remaining part of the integrand of (1), following Smirnov (1961),

$$\begin{aligned} &u^{2n_1+L}Y^*_{LM}(\Omega_u)v^{2n_2+L}Y_{LM}(\Omega_v) \\ &\quad =(-1)^M \sum_{\substack{NL'M'\lambda \\ nl'm'm_\lambda}} C(n_1L, n_2L, nl', NL')(LM, L-M|\lambda 0) \\ &\qquad \times(L'M', l'm'|\lambda m_\lambda)\dot{B}_\lambda(n_1L, n_2L, nl', NL') \\ &\qquad \times P^{2n+l'}Q^{2N+L'}Y_{L'M'}(\Omega_Q)Y_{l'm'}(\Omega_P) \end{aligned} \tag{6}$$

where $(LM, L-M|\lambda 0)$ and $(L'M', l'm'|\lambda m_\lambda)$ are Clebsch-Gordan coefficients and

$$\begin{aligned} &C(n_1L_1, n_2L_2, nl, NL) \\ &\quad =(-1)^{N+n-n_1-n_2} \\ &\qquad \times[n_1!\, n_2!\,(2n_1+2L+1)!!\,(2n_2+2L+1)!!]^{1/2} \\ &\qquad \times[2^{n_1+L_1+n_2+L_2-n-l-N-L}n!\,N!\,(2n+2l+1)!!\,(2N+2L+1)!!]^{-1/2} \end{aligned}$$

Taking (5) and (6) into account, as well as the fact that the Jacobian for the transformation (4) is $2D^{1/2}/(1+D)$, we write (1) as

$$\begin{aligned} &I_{w_1,w_2,L}(p, q, s) \\ &\quad =(-1)^M\left[\frac{2D^{1/2}}{1+D}\right](p^{w_1+3}s^{w_2+3})^{-1/2} \\ &\qquad \times \sum_{\substack{NL'M'\lambda \\ nl'm'm_\lambda}} C(n_1L, n_2L, nl', NL')(LM, L-M|\lambda 0)(L'M', l'm'|\lambda m_\lambda) \\ &\qquad \times\int P^{2n+l'+2}Q^{2N+L'+2}\exp(-aP^2-bQ^2) \\ &\qquad \times Y_{L'M'}(\Omega_Q)Y_{l'm'}(\Omega_P)\, dP\, dQ\, d\Omega_P\, d\Omega_Q \end{aligned} \tag{7}$$

Since the variables separate, the right-hand side of (7) can be easily evaluated. The integration with respect to the angle variables yields: $4\pi\delta_{L',0}\delta_{M',0}\delta_{l',0}\delta_{m',0}$, which reduces the number of summation indices in (7) to two (viz. n and N). The remaining radial integrals are Laplace transforms (see, e.g., Erdelyi *et al.*, 1954) which can be expressed in terms of hypergeometric functions. After integration, we obtain

$$\begin{aligned} I_{w_1,w_2,L}(p,q,s) &= (-1)^L[L]^{-1}\pi^2(p^{2n_1+L+3}s^{2n_2+L+3})^{-1/2} \\ &\times \sum_{n,N} 2^{-(n+N+2)}\{2^{-(n+N+2)}(1+D)^{n+N+2}/D^{n+1}\} \\ &\times (2n+1)!!(2N+1)!!C(n_1L, n_2L, n0, N0)B_0(n_1L, n_2L, n0, N0) \\ &\times {}_2F_1\left(n+\frac{3}{2}, \beta; \beta; -\frac{q}{2(ps)^{1/2}}\right){}_2F_1\left(N+\frac{3}{2}, \beta; \beta; \frac{q}{2(ps)^{1/2}}\right) \end{aligned} \tag{8}$$

where β is a dummy parameter and $[L]=(2L+1)^{1/2}$.

3. THE IDENTITY

In the preceding section, we have shown how the integral (1) can be evaluated by two different methods to yield the results given in (3) and (8). Equating these two results, we arrive at the identity:

$$\begin{aligned} &\sum_{n,N} \{2^{-(n+N+2)}(1+D)^{n+N+2}/D^{n+1}\}2^{-(n+N)}(2n+1)!!(2N+1)!! \\ &\quad \times C(n_1L, n_2L, n0, N0)B_0(n_1L, n_2L, n0, N0) \\ &\quad \times {}_2F_1\left(n+\frac{3}{2}, \beta; \beta; -\frac{q}{2(ps)^{1/2}}\right){}_2F_1\left(N+\frac{3}{2}, \beta; \beta; \frac{q}{2(ps)^{1/2}}\right) \\ &= (-1)^L[L]q^L[2^{n_1+n_2+L}(ps)^{L/2}(2L+1)!!]^{-1} \\ &\quad \times (2n_1+2L+1)!!\,(2n_2+2L+1)!! \\ &\quad \times {}_2F_1\left(n_1+L+\frac{3}{2}, n_2+L+\frac{3}{2}; L+\frac{3}{2}; \frac{q^2}{4ps}\right) \end{aligned} \tag{9}$$

which is our result.

For the equal-mass case, the quantity within curly brackets in (9) reduces to 1 and the generalized TMB, $B_0(n_1L, n_2L, n0, N0)$, should be replaced by the TMB: $\langle n_1L, n_2L, 0|n0, N0, 0\rangle$.

It is interesting to note that this identity is solely responsible for the nucleon exchange integrals, encountered by Kudeyarov et al. (1971) in their study of the charge and magnetic form factors of ^{6}Li, reducing to their corresponding normalization integrals, at zero momentum transfer.

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